

AN EMBEDDING CONSTANT FOR THE HARDY SPACE OF DIRICHLET SERIES

OLE FREDRIK BREVIG

ABSTRACT. A new and simple proof of the embedding of the Hardy–Hilbert space of Dirichlet series into a conformally invariant Hardy space of the half-plane is presented, and the optimal constant of the embedding is computed.

Let \mathcal{H}^2 denote the Hardy–Hilbert space of Dirichlet series, $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, with square summable coefficients, and set

$$\|f\|_{\mathcal{H}^2} := \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{\frac{1}{2}}.$$

Using the Cauchy–Schwarz inequality, we find that a Dirichlet series $f \in \mathcal{H}^2$ is absolutely convergent in the half-plane $\mathbb{C}_{1/2} := \{s : \Re(s) > 1/2\}$. To see that $\mathbb{C}_{1/2}$ is the largest half-plane of convergence for \mathcal{H}^2 , consider $f(s) = \zeta(1/2 + \varepsilon + s)$, where ζ denotes the Riemann zeta function and $\varepsilon > 0$.

When studying function and operator theoretic properties of \mathcal{H}^2 , it has proven fruitful to employ the embedding of \mathcal{H}^2 into the conformally invariant Hardy space of $\mathbb{C}_{1/2}$ (see e.g. [6, Sec. 9]). The embedding inequality takes on the form

$$(1) \quad \|f\|_{H^2_i} := \left(\frac{1}{\pi} \int_{-\infty}^{\infty} |f(1/2 + it)|^2 \frac{dt}{1+t^2} \right)^{\frac{1}{2}} \leq C \|f\|_{\mathcal{H}^2}.$$

Observe that the embedding inequality (1) implies that Dirichlet series in \mathcal{H}^2 are locally L^2 -integrable on the line $\Re(s) = 1/2$. Indeed, the proofs of (1) in the literature go via the local (but equivalent) formulation

$$(2) \quad \sup_{\tau \in \mathbb{R}} \left(\int_{\tau}^{\tau+1} |f(1/2 + it)|^2 dt \right)^{\frac{1}{2}} \leq \tilde{C} \|f\|_{\mathcal{H}^2}.$$

To prove (2), one can use a general Hilbert–type inequality due to Montgomery and Vaughan [3] or a version of the classical Plancherel–Polya inequality [2, Thm. 4.11]. It is also possible to give Fourier analytic proofs of (2), the reader is referred to [4, pp. 36–37] and [5, Sec. 1.4.4]. It should be pointed out that these proofs do not give a precise value for either of the constants C and \tilde{C} .

This note contains a new and simple proof of (1), which additionally identifies the optimal constant C . The proof is based on the observation that the H^2_i -norm of a Dirichlet series is associated to a Hilbert–type bilinear form which is easy to estimate precisely.

Date: June 13, 2016.

2010 *Mathematics Subject Classification.* Primary 30B50. Secondary 15A63.

The author is supported by Grant 227768 of the Research Council of Norway.

Theorem. Suppose that $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is in \mathcal{H}^2 . Then

$$\left(\frac{1}{\pi} \int_{-\infty}^{\infty} |f(1/2 + it)|^2 \frac{dt}{1+t^2} \right)^{\frac{1}{2}} < \sqrt{2} \|f\|_{\mathcal{H}^2},$$

and the constant $\sqrt{2}$ is optimal.

Proof. Let x be a positive real number. We begin by computing

$$I(x) := \frac{1}{\pi} \int_{-\infty}^{\infty} x^{it} \frac{dt}{1+t^2} = \frac{1}{\pi} \int_{-\infty}^{\infty} \cos(|\log x| t) \frac{dt}{1+t^2} = e^{-|\log x|} = \frac{1}{\max(x, 1/x)}.$$

Expanding $|f(1/2 + it)|^2$, we find that

$$(3) \quad \|f\|_{H_1^2}^2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m \bar{a}_n}{\sqrt{mn}} I(n/m) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m \bar{a}_n \frac{\sqrt{mn}}{[\max(m, n)]^2}.$$

The identity (3) will serve as the starting point for both the proof of the inequality $\|f\|_{H_1^2} < \sqrt{2} \|f\|_{\mathcal{H}^2}$, and for the proof that $\sqrt{2}$ cannot be improved.

Let us first consider the Hilbert-type (see [1, Ch. IX]) bilinear form associated to (3). Given sequences $a, b \in \ell^2$, we want to estimate

$$B(a, b) := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n \frac{\sqrt{mn}}{[\max(m, n)]^2}.$$

By the Cauchy–Schwarz inequality, we find that

$$|B(a, b)| \leq \left(\sum_{m=1}^{\infty} |a_m|^2 \sum_{n=1}^{\infty} \frac{m}{[\max(m, n)]^2} \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} |b_n|^2 \sum_{m=1}^{\infty} \frac{n}{[\max(m, n)]^2} \right)^{\frac{1}{2}}.$$

Then $|B(a, b)| < 2 \|a\|_{\ell^2} \|b\|_{\ell^2}$, since

$$\sum_{n=1}^{\infty} \frac{m}{[\max(m, n)]^2} = \sum_{n=1}^m \frac{m}{m^2} + \sum_{n=m+1}^{\infty} \frac{m}{n^2} < 1 + m \int_m^{\infty} \frac{dx}{x^2} = 2.$$

Setting $b = \bar{a}$, we obtain the desired inequality $\|f\|_{H_1^2} < \sqrt{2} \|f\|_{\mathcal{H}^2}$.

For the optimality of $\sqrt{2}$, we again let $f(s) = \zeta(1/2 + \varepsilon + s)$ for some $\varepsilon > 0$. Clearly, $\|f\|_{\mathcal{H}^2}^2 = \zeta(1 + 2\varepsilon)$. We insert f into (3) and estimate the inner sums using integrals, which yields

$$\begin{aligned} \|f\|_{H_1^2}^2 &= \sum_{m=1}^{\infty} m^{-\varepsilon} \left(\frac{1}{m^2} \sum_{n=1}^m n^{-\varepsilon} + \sum_{n=m+1}^{\infty} \frac{n^{-\varepsilon}}{n^2} \right) \\ &> \sum_{m=1}^{\infty} m^{-\varepsilon} \left(\frac{1}{m^2} \frac{m^{1-\varepsilon} - 1}{1 - \varepsilon} + \frac{(m+1)^{-1-\varepsilon}}{1 + \varepsilon} \right) \\ &> \frac{\zeta(1 + 2\varepsilon) - \zeta(2 + \varepsilon)}{1 - \varepsilon} + \frac{\zeta(1 + 2\varepsilon) - 1}{1 + \varepsilon}. \end{aligned}$$

Letting $\varepsilon \rightarrow 0^+$, we conclude that if $\|f\|_{H_1^2} \leq C \|f\|_{\mathcal{H}^2}$, then $C^2 \geq 2$. \square

REFERENCES

1. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, second ed., Cambridge University Press, 1952.
2. H. Hedenmalm, P. Lindqvist, and K. Seip, *A Hilbert space of Dirichlet series and systems of dilated functions in $L^2(0, 1)$* , Duke Math. J. **86** (1997), no. 1, 1–37.
3. H. L. Montgomery and R. C. Vaughan, *Hilbert's inequality*, J. London Math. Soc. (2) **8** (1974), 73–82.
4. J.-F. Olsen and E. Saksman, *On the boundary behaviour of the Hardy spaces of Dirichlet series and a frame bound estimate*, J. Reine Angew. Math. **663** (2012), 33–66.
5. H. Queffélec and M. Queffélec, *Diophantine approximation and Dirichlet series*, Harish-Chandra Research Institute Lecture Notes, vol. 2, Hindustan Book Agency, New Delhi, 2013.
6. H. Queffélec and K. Seip, *Approximation numbers of composition operators on the H^2 space of Dirichlet series*, J. Funct. Anal. **268** (2015), no. 6, 1612–1648.

DEPARTMENT OF MATHEMATICAL SCIENCES, NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY (NTNU), NO-7491 TRONDHEIM, NORWAY

E-mail address: `ole.brevig@math.ntnu.no`